Journal of Geometry and Physics 11 (1993) 553–557
North-Holland

JOURNAL OF GEOMETRY AND PHYSICS

Geodesics on loop spaces

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The space of loops smoothly embedded into a Riemannian manifold, being a principal fibre bundle with structure group Diff S^1, is investigated from a Kaluza–Klein type point of view. In particular, the Levi-Civita connection for the natural Diff S^1-invariant metric on this loop space is calculated and the corresponding horizontal geodesics (the analogue of classical free motion of point particles) are characterized. Finally, an explicit solution is given in the case of loops in $\mathbb{R}^3$.

Keywords: *loop spaces, manifolds of maps, Kaluza–Klein theories, Levi-Civita connection, (horizontal) geodesics*
1991 MSC: 58D10, 58D15 (83E15, 83E30)

The classical free motion of point particles moving in a finite dimensional Riemannian manifold $(M, g = \langle\ ,\ \rangle)$ is described by geodesics with respect to the Levi-Civita connection ∇ of g. In order to examine the analogue for the space of 'unparametrized' loops in M, the configuration space of string theory, we consider the space $E = E(S^1, M)$ of smooth embeddings γ: $S^1 \to M$ ($\dim M > 1$). Equipped with Whitney's C^∞-topology given by uniform convergence of all derivatives, E is an open subset of the smooth Fréchet manifold $C^\infty(S^1, M)$ (cf. ref. [1,2,4–6,8]). By a generalization of Whitney's embedding theorem [5], it is even dense in $C^k(S^1, M)$ for $0 \le k \le \infty$, if $\dim M \ge 3$. Thus, although having excluded (for the moment) all kinds of degenerate loops like trivial or self-intersecting loops, we may expect to get some information also for these cases.

Reparametrization of loops amounts to a right action of $\mathrm{Diff}\, S^1$ on E via *pull-back*:

$$E \times \mathrm{Diff}\, S^1 \longrightarrow E,\ (\gamma, \varphi) \longmapsto \varphi^*(\gamma) = \gamma \circ \varphi.$$

This makes E into a principal-Diff S^1-fibre bundle over the space of our primary interest, which is the space $U = U(S^1, M)$ of all submanifolds of M diffeomorphic to S^1 [1,6]. In addition, the total space E of this bundle comes equipped with a canonical Diff S^1-invariant Riemannian metric $\widetilde{G}$ (defined below), which induces a connection on E and a Riemannian metric $\check{G}$ on U. As in the Kaluza-Klein approach (cf. ref. [3]), the projection of horizontal geodesics on E with

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respect to the Levi-Civita connection $\widetilde{\nabla}$ for $\widetilde{G}$ onto U are geodesics of $\check{G}$. The topic of this paper is a closer description of these concepts (cf. [7] for details and further results).

To start, let us first introduce the *Ω-lift* which assigns for two manifolds M_1 and M_2 to each $\phi \in \mathrm{C}^\infty(M_1, M_2)$ the map

$$\Omega\phi\colon \mathrm{C}^\infty(\mathrm{S}^1, M_1) \longrightarrow \mathrm{C}^\infty(\mathrm{S}^1, M_2), \gamma \longmapsto \Omega\phi(\gamma) = \phi \circ \gamma.$$

It plays an important rôle in concrete calculations, as will become clear in the sequel. For the moment, just observe that the natural left action of Diff M on E is given via the Ω-lift.

The tangent space $\mathrm{T}_\gamma E$ of E at the point γ is the space of vector fields along γ:

$$\mathrm{T}_\gamma E = \{\widehat{X} \in \mathrm{C}^\infty(\mathrm{S}^1, TM) \mid \Omega\tau_M(\widehat{X}) = \tau_M \circ \widehat{X} = \gamma\},$$

where τ_M is the natural projection of the tangent bundle $\tau_M\colon TM \to M$. (Example: γ', the vector field along γ, which assigns to each point the tangent vector to the curve γ at this point.)

For each tangent vector $\widehat{X}_\gamma \in \mathrm{T}_\gamma E$ there exists a vector field $X \in \mathcal{X}(M)$, called *associated vector field*, such that $\widehat{X}_\gamma = \Omega X(\gamma)$. Tensor fields on E are therefore completely determined by their values on Ω-lifts of vector fields on M. Furthermore, although there is no general existence and uniqueness theorem for integral curves and flows of vector fields on Fréchet manifolds, each vector field $\Omega X \in \mathcal{X}(E)$ has a unique local flow on E at γ, namely the Ω-lift of the flow of X. For the Lie derivative $\widehat{L}_{\Omega X}$ defined via pull-back the following holds:

$$\begin{aligned} [\Omega X, \Omega Y] = \widehat{L}_{\Omega X}\Omega Y &= \Omega\,(L_X Y) = \Omega([X, Y]) \\ \widehat{L}_{\Omega X}\Omega f &= \Omega\,(L_X f) \end{aligned} \tag{1}$$

for all $X, Y \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$.

As mentioned before, E is a principal fibre bundle with structure group Diff S^1. The vertical subspace $\mathcal{V}_\gamma \subset \mathrm{T}_\gamma E$ is the space of those vector fields along γ that are tangent to $\gamma(\mathrm{S}^1)$. In particular, the fundamental vector field ${}_E\xi \in \Gamma\mathcal{V} \subset \mathcal{X}(E)$ of the right action of Diff S^1 on E generated by $\xi \in \mathcal{L}(\mathrm{Diff}\,\mathrm{S}^1) \simeq \mathcal{X}(\mathrm{S}^1)$ is given by

$${}_E\xi(\gamma) = (\gamma_*\xi) \circ \gamma.$$

(Example: $\gamma' = (\gamma_*\partial_s) \circ \gamma$, the fundamental vector field corresponding to the 'generator of constant rotations' $\partial_s \in \mathcal{X}(\mathrm{S}^1)$.)

Any smooth function $\mu\colon E \to \mathrm{Vol}(\mathrm{S}^1)$ which assigns to $\gamma \in E$ a volume form μ_γ on S^1 with corresponding measure $\mathrm{d}\mu_\gamma$, induces a weak Riemannian metric on E:

$$G(\gamma)\,(\widehat{X}_\gamma, \widehat{Y}_\gamma) = \int_{\mathrm{S}^1} \mathrm{d}\mu_\gamma(s)\,\langle \widehat{X}_\gamma(s), \widehat{Y}_\gamma(s)\rangle_{\gamma(s)} = \int_{\mathrm{S}^1} \mu_\gamma \Omega(\langle X, Y\rangle)(\gamma)$$

for all $\widehat{X}_\gamma = \Omega X(\gamma)$, $\widehat{Y}_\gamma = \Omega Y(\gamma) \in \mathrm{T}_\gamma E$.

The case concerning us here is the map μ which assigns to each $\gamma \in E$ the Riemannian volume μ_γ corresponding to the metric $\gamma^* g$ on S^1. This leads to the natural Diff S^1-invariant metric

$$\widetilde{G}(\gamma)\,(\widehat{X}_\gamma, \widehat{Y}_\gamma) = \int_{\mathrm{S}^1} \mathrm{d}s\, \sqrt{\langle \gamma'(s), \gamma'(s)\rangle_{\gamma(s)}}\, \langle \widehat{X}_\gamma(s), \widehat{Y}_\gamma(s)\rangle_{\gamma(s)}\,.$$

on E, which is also invariant under the left action of the subgroup $\mathrm{Iso}(M, g) \subset \mathrm{Diff}\, M$ of isometries of g. Because of its Diff S^1-invariance, $\widetilde{G}$ induces a connection on E – assigning to each $\gamma \in E$ the horizontal subspace $\mathcal{H}_\gamma = \mathcal{V}_\gamma^\perp$ (the orthogonal complement with respect to $\widetilde{G}$), which is the space of sections in the normal bundle over the submanifold $\gamma(\mathrm{S}^1)$ in M – and a metric $\check{G}$ on U – $\check{G}(\check{\gamma})\,(\check{X}_{\check{\gamma}}, \check{Y}_{\check{\gamma}}) = \widetilde{G}(\gamma)\,(\overline{X}_\gamma, \overline{Y}_\gamma)$, where $\overline{X}_\gamma, \overline{Y}_\gamma \in \mathcal{H}_\gamma$ are the horizontal lifts of $\check{X}_{\check{\gamma}}, \check{Y}_{\check{\gamma}} \in \mathrm{T}_{\check{\gamma}} U$ for some $\gamma \in \pi^{-1}(\check{\gamma})$.

We now sketch the proof of existence and uniqueness of the Levi-Civita connection $\widetilde{\nabla}$ corresponding to $\widetilde{G}$. The problem is that for infinite dimensional Riemannian manifolds the six-term formula of Levi-Civita

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \tfrac{1}{2}\{&L_X \langle Y, Z\rangle + L_Y \langle X, Z\rangle - L_Z \langle X, Y\rangle \\ &- \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle\}\end{aligned} \tag{2}$$

merely guarantees uniqueness of ∇, but tells us nothing about existence. It is therefore useful to have a brief look at the *trivial metric* $\widehat{G}$, induced by the constant function $\mu_\gamma = \mathrm{d}s$, because an explicit definition of the corresponding Levi-Civita connection $\widehat{\nabla}$ can be given, for instance, via parallel transport. Moreover (2) yields by (1)

$$\widehat{\nabla}_{\Omega X} \Omega Y = \Omega\,(\nabla_X Y), \qquad X, Y \in \mathcal{X}(M). \tag{3}$$

So geodesics with respect to $\widehat{\nabla}$ on E are simply 'collections' of those of ∇ on M, hence they exist and are uniquely determined by their initial values.

Once having $\widehat{\nabla}$ at hand, one can use (2) to prove existence and uniqueness of the Levi-Civita connection $\widetilde{\nabla}$, since it is uniquely determined by the the symmetric type $(2, TE)$ tensor field $\widetilde{S}(\widehat{X}, \widehat{Y}) = \widetilde{\nabla}_{\widehat{X}} \widehat{Y} - \widehat{\nabla}_{\widehat{X}} \widehat{Y}$. In contrast to the calculation for $\widehat{\nabla}$ resulting in (3), the volume form μ_γ in the formula for $\widetilde{G}$ is now a (non-constant) function of γ. So (2) contains terms like

$$(\widehat{L}_{\Omega X} \widetilde{G}(\Omega Y, \Omega Z))\,(\gamma) = \int_{\mathrm{S}^1} (\widehat{L}_{\Omega X} \mu)_\gamma \Omega\,\langle Y, Z\rangle\,(\gamma) + \int_{\mathrm{S}^1} \mu_\gamma \Omega\,(L_X \langle Y, Z\rangle)(\gamma).$$

(For $\widehat{G}$, only the analogue of the second term appears). Since $\widetilde{G}$ and $\widetilde{S}$ are tensor fields on E and $\widetilde{G}$ is nondegenerate, using (3), one finally arrives at

$$\widetilde{S}\,(\Omega X, \Omega Y)\,(\gamma) = \tfrac{1}{2}\{\langle \tau', \nabla_{\tau'} X\rangle\, Y + \langle \tau', \nabla_{\tau'} Y\rangle\, X + \nabla_{\tau'}(\langle X, Y\rangle\, \tau')\}$$

with $\tau'(s) = \gamma'(s)/\|\gamma'(s)\|$ ($\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$). The corresponding geodesic equation $\widetilde{\nabla}_{\dot\gamma_t}\dot\gamma_t = 0$ leads to the nonlinear PDE

$$2\frac{\mathrm{D}}{\partial t}\Big(\|\gamma_t'(s)\|\,\dot\gamma_t(s)\Big) + \frac{\mathrm{D}}{\partial s}\Big(\|\dot\gamma_t(s)\|^2\,\tau_t'(s)\Big) = 0 \tag{4}$$

for smooth curves γ_t in E, where $\gamma_t' = \partial\gamma_t/\partial s$, resp. $\dot\gamma_t = \partial\gamma_t/\partial t$ denote the tangent vector field along the curve $\gamma_t(s)$ for fixed t, resp. for fixed $s \in \mathrm{S}^1$ and $\mathrm{D}/\partial s$ resp. $\mathrm{D}/\partial t$ the covariant derivative of the Levi-Civita connection of g along the corresponding curves in M. The solutions of (4) are the stationary points of the action functional

$$S = \int_{t_0}^{t_1} \mathrm{d}t\,\widetilde{G}(\gamma_t)\,(\dot\gamma_t,\dot\gamma_t). \tag{5}$$

As already mentioned, our aim is the description of horizontal geodesics on E, i.e., those with $\langle\dot\gamma_t(s),\gamma_t'(s)\rangle = 0$ for all t and $s \in \mathrm{S}^1$, because they project onto geodesics on U.

Observe that geodesics on E which start horizontal remain horizontal for all 'time', since from the Diff S^1-invariance and the fact that all geodesics are 'constantly parametrized', one obtains the conservation laws

$$\frac{\partial}{\partial t}(\|\gamma_t'\|\,\langle\dot\gamma_t,\gamma_t'\rangle) = 0 = \frac{\partial}{\partial t}(\|\gamma_t'\|\,\langle\dot\gamma_t,\dot\gamma_t\rangle).$$

(The second one holds for horizontal γ_t only.) Moreover, additional conservation laws follow from the invariance of $\widetilde{G}$ under the left action of isometries of g, whereas uniqueness of $\widetilde{\nabla}$ implies that the Ω-lifts of isometries of g map geodesics onto geodesics.

Since we have (at the moment) no existence and uniqueness theorem for geodesics on E in general, we shall give here a solution of (4) on $E(\mathrm{S}^1,\mathbb{R}^3)$ for the following initial values ($R_0 > 0$, $s \in [0,2\pi]$):

$$\begin{aligned}\gamma_0(s) &= R_0(0,\cos s,\sin s),\\ \dot\gamma_0(s) &= (\dot X_0,\dot R_0\cos s,\dot R_0\sin s).\end{aligned}$$

If we had a uniqueness theorem for geodesics, we could conclude that the solution must have the same invariance under isometries, i.e. rotations around the 1-axis, as the initial data. So it would have to be of the form

$$\gamma_t(s) = (x(t), r(t)\cos s, r(t)\sin s). \tag{6}$$

Taking (6), which always gives horizontal geodesics, as an ansatz, (4) reduces to two ordinary differential equations for $x(t)$ and $r(t)$, with solutions

$$r(t) = R_0\left(1 + \frac{3\dot R_0}{2R_0}t\right)^{2/3} \quad \text{for } \dot X_0 = 0,$$

$$r(x) = R_0 \left[\frac{1+b^2}{4} \left(\frac{x}{R_0} + \frac{2b}{1+b^2} \right)^2 + \frac{1}{1+b^2} \right] \text{ for } \dot{X}_0 \neq 0,$$

with $b = \dot{R}_0/\dot{X}_0$, where we have eliminated t for x in the second case to get an impression of the shape of the geodesic as a surface in $\mathbb{R}^3$.

This example shows that there are plane circular loops $\gamma^{(0)}, \gamma^{(1)} \in E(S^1, \mathbb{R}^3)$, such that there exist(s) no/a unique/two geodesic(s) of the form (6) with $\gamma^{(0)} = \gamma_0$ and $\gamma^{(1)} = \gamma_1$. In the third case, only the solution with the smaller absolute value of b minimizes the action functional (5).

I should like to thank E. Meinrenken for careful reading of the manuscript.

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